Are solutions of reaction-diffusion equations asymptotically 1D ?

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Mostly Maximum Principle
4^{th} edition

Cortona

Collaboration with François Hamel

Main question

$$\partial_t u - \Delta u = f(u) \qquad t > 0,\ x \in \mathbb{R}^N$$

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Let f be Allen-Cahn: $f(u) = u(1-u)(1/2 - u)$.
Let $0 < u(x) < 1$ be a solution satisfying $\partial_1 u > 0$.
Then u is 1-D, i.e. $u(x) = \Psi(x \cdot e)$ for some e, at least if $N \leq 8$

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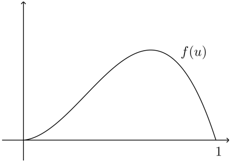
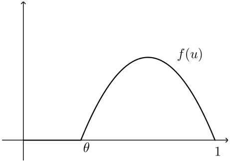
In our case 1 is more attractive than 0

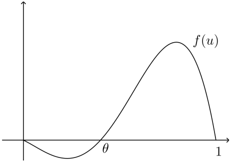
$f(u)$ of one of the following types:



monostable



combustion



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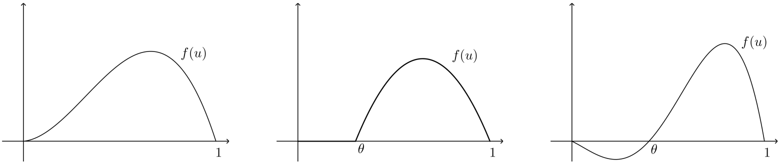

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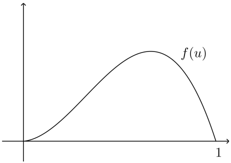

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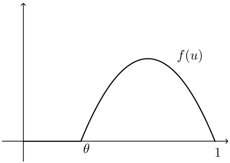

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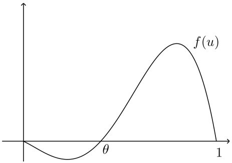

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If U contains a sufficiently large ball $\implies$ invasion holds:
$u(t,\cdot) \to 1$ locally uniformly as $t \to +\infty$

What does "becoming locally planar" mean ?

$$\begin{aligned} \partial_t u - \Delta u &= f(u) \qquad t > 0,\ x \in \mathbb{R}^N \\ u(0,x) &= \mathbb{1}_U(x) \qquad x \in \mathbb{R}^N \end{aligned}$$

U is a generic subset of $\mathbb{R}^N$.

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Definition

The Ω-limit set of a solution u is

$$\Omega(u) := \big\{ \psi \in L^\infty(\mathbb{R}^N) \ :\ u(t_n, x_n + \cdot) \to \psi \text{ in } L^\infty_{loc}(\mathbb{R}^N) \text{ as } n \to +\infty, \\ \text{for some sequences } t_n \to +\infty \text{ and } (x_n)_{n\in\mathbb{N}} \text{ in } \mathbb{R}^N \big\}$$

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u is asymptotically locally planar if $\Omega(u)$ contains only $1D$ functions, i.e.

$$\psi(x) = \Psi(x \cdot e) \quad \text{for some } e \in \mathbb{R}^N$$

Known cases where u is asymptotically locally planar

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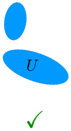

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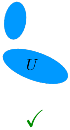

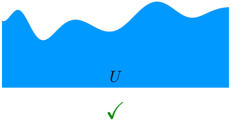

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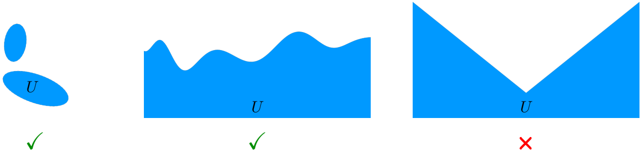

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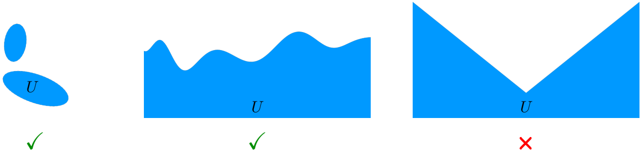

Conjecture

If U is at finite Hausdorff distance from a convex set then the solution u is asymptotically locally planar

Hausdorff distance: $$d_{\mathcal{H}}(U,V) := \sup_{x\in U} \mathrm{dist}(x,V) \vee \sup_{y\in V} \mathrm{dist}(y,U)$$

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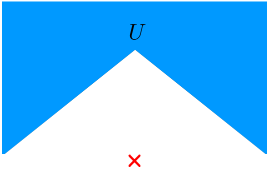

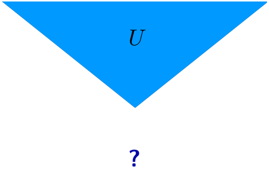

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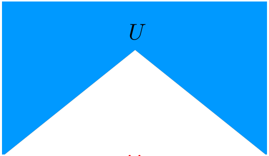

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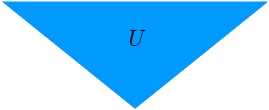

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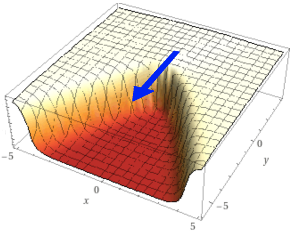

$u \to$ V-shaped front

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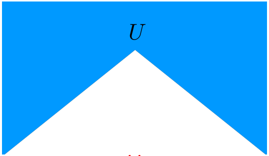

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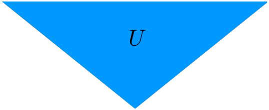

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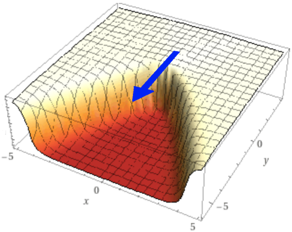

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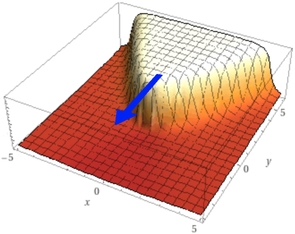

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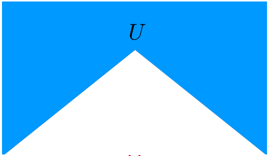

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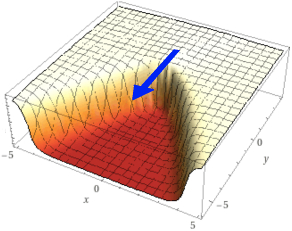

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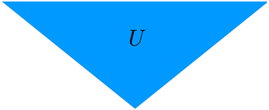

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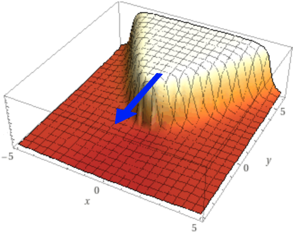

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Spreading set result: $\{u(t,\cdot) > \lambda\} \sim U + B_{c^* t} \qquad \forall \lambda \in (0,1)$

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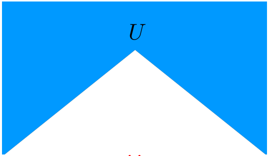

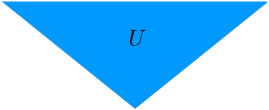

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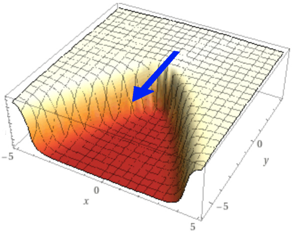

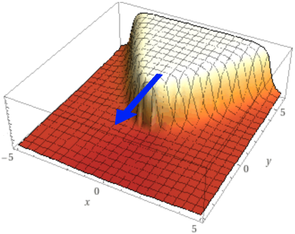

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Not a proof !! because "$\sim$" means "up to $o(t)$"

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Assume that f is strongly-KPP. If U is at finite Hausdorff distance from a convex set and is uniformly smooth, then $\Omega(u)$ consists exclusively of $1D$ functions

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Main tool of the proof

Extension of Jones' argument to initial data not compactly supported

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Under the assumptions of the Theorem, all-but-at-most-1 the eigenvalues of $D^2u(t,x)$ tend to 0 as $t \to +\infty$ uniformly in $x \in \mathbb{R}^N$ $\implies$

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Open questions

- Is the conjecture true beyond the Fisher-KPP case ?
- Under the assumptions of the theorem, does $\Omega(u)$ consists exclusively of constant functions and (translations of) the critical front ?

Grazie per la vostra attenzione !!